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ORIGINAL PAPER

Approximate series solution of fourth-order boundary value problems using decomposition method with Green's function

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Abstract This paper proposes a new efficient approach for obtaining approximate series solutions to fourth-order two-point boundary value problems. The proposed approach depends on constructing Green's function and Adomian decomposition method (ADM). Unlike existing methods like ADM or modified ADM, the proposed approach avoids solving a sequence of nonlinear equations for the undetermined coefficients. In fact, the proposed method gives a direct recursive scheme for obtaining approximations of the solution with easily computable components. We also discuss the convergence and error analysis of the proposed scheme. Moreover, several numerical examples are included to demonstrate the accuracy, applicability, and generality of the proposed approach. The numerical results reveal that the proposed method is very effective and simple.

Keywords Boundary value problem · Adomian decomposition method · Modified Adomian decomposition method · Approximations · Green's function

Mathematics Subject Classification 34B05 · 34B15 · 34B16 · 34B18 · 34B27

1 Introduction

Accurate and fast numerical solution of two-point boundary value ordinary differential equations is necessary in many important scientific and engineering applications, e.g.,

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heat and mass transfer within porous catalyst particle [1], oxygen diffusion in cells [2], astrophysics, hydrodynamic and hydromagnetics stability, boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology. Fourth-order BVPs for ordinary differential equation have attracted much attention in recent years. Such problems arise in the study of mathematical modeling of viscoelastic and inelastic flows [3], deformation of beams [4] and plate deflection theory [5,6].

The aim of this article is to propose an efficient technique to solve a general linear as well as nonlinear fourth-order BVPs. The proposed technique is based on the Adomian decomposition method and the Green's function technique, here we transform fourth-order BVPs into an equivalent Fredholm-integral equation before establishing the recursive scheme for the solution. Consider the following fourth-order BVPs [7–10]:

$$u^{(iv)}(x) = g(x) + f(x, u(x), u'(x), u''(x), u'''(x)), \quad x \in J = [0, b], \tag{1.1}$$

subject to the boundary conditions

$$u(0) = \alpha_1,\; u'(0) = \alpha_2,\; u(b) = \alpha_3,\; u'(b) = \alpha_4, \tag{1.2}$$

where α_i, $i = 1, 2, 3, 4$ are finite real constants. We assume that $f(x, u, u', u'', u''')$ is continuous on $D = \{[0, b] \times \mathbb{R}^4\}$ and is not identically zero.

There is considerable literature on the numerical treatment of fourth-order BVPs [5,7–17] and the references cited therein. Various efficient numerical techniques have been used to deal with such BVPs, such as finite difference method (FDM) [5,17]. Although, these techniques are very efficient and have many advantages, but a huge amount of computational work is needed which combines some root-finding techniques for obtaining accurate numerical solution especially for nonlinear fourth-order BVPs. Recently, some newly developed numerical-approximate methods have also been applied for the solution of BVPs (1.1), (1.2), such as Adomian decomposition method (ADM) and modified ADM (MADM) [7,8,18], homotopy perturbation method (HPM), homotopy analysis method (HAM) [10,12] and the differential transform method (DTM) [15,16]. The variational iteration method (VIM) and modified VIM were also used in [13,14]. However, methods like VIM or modified VIM fail to solve the equation when the nonlinear function is of the form e^u, $\ln(u)$, $\sin u$, $\sinh u$... etc., see Wazwaz and Rach [19] for more details. Nevertheless, applications of VIM for solving nonlinear problems are very restrictive.

Furthermore, note that solving nonlinear fourth-order BVPs using ADM or MADM is always a computationally involved task. Since, it requires the computation of unknown constants in a sequence of nonlinear algebraic or difficult transcendental system of equations which increases the computational work (see [20]). Moreover, in some cases, the unknown constants may not be uniquely determined and this may be the major disadvantage of these methods for solving fourth-order BVPs.

The objective of this work is to propose a modification of the ADM which combines with Green's function to overcome the difficulties occurring in the ADM or MADM for solving two-point fourth-order BVPs (1.1), (1.2). This methodology relies on constructing Green's function before establishing the recursive scheme for the solution components. Unlike ADM or MADM, the proposed method avoids solving a sequence of transcendental equations for the undetermined coefficients. In other words, we develop a direct scheme for obtaining approximate series solutions. The approximations of the solution are obtained in the form of series with easily calculable components. For the completeness, the convergence and error analysis of the proposed scheme is supplemented. Moreover, several numerical examples are included to demonstrate the accuracy, applicability, and generality of the proposed scheme.

2 Review of classical ADM/MADM

In this section, we briefly describe ADM or MADM for solving fourth-order BVPs (1.1), (1.2). It is well-known that ADM allows us to solve both nonlinear IVPs and BVPs without unphysical restrictive assumptions such as linearization, discretization, perturbation and guessing the initial term or a set of basis function. Recently, many researchers [7,8,11,20–32] have applied the ADM or MADM for solving the different scientific models. According to the ADM, BVPs (1.1), (1.2) can be written in operator form as

$$\mathcal{L}u(x) = g(x) + Nu(x), \quad x \in J, \tag{2.1}$$

subject to the boundary conditions

$$u(0) = \alpha_1, \; u'(0) = \alpha_2, \; u(b) = \alpha_3, \; u'(b) = \alpha_4, \tag{2.2}$$

where $\mathcal{L} = \frac{d^4}{dx^4}$ is a fourth-order linear differential operator, $Nu(x) = f(x, u, u', u'', u''')$ denotes the nonlinear function and $g(x)$ is known function. The inverse operator of $\mathcal{L}$ is defined as

$$\mathcal{L}^{-1}[\cdot] = \int_0^x \int_0^x \int_0^x \int_0^x [\cdot] dx dx dx dx. \tag{2.3}$$

By operating $\mathcal{L}^{-1}[\cdot]$ on both sides of (2.1) and using the conditions $u(0) = \alpha_1$ and $u'(0) = \alpha_2$, we obtain

$$u(x) = \alpha_1 + \alpha_2 x + c_1 x^2 + c_2 x^3 + \mathcal{L}^{-1}[g(x) + Nu(x)], \tag{2.4}$$

where $c_1 = \frac{u''(0)}{2}$ and $c_2 = \frac{u''(0)}{6}$ are unknown constants be determined.

Then the solution $u(x)$ and the nonlinear function $Nu(x)$ are decomposed by

$$u(x) = \sum_{j=0}^{\infty} u_j(x) \quad \text{and} \quad Nu(x) = \sum_{j=0}^{\infty} A_j, \tag{2.5}$$

where $A_j, j = 0, 1, \ldots$ are Adomian's polynomials which can be calculated for various classes of nonlinear functions with the formula given in [31] as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\right]_{\lambda=0}, \; n = 0, 1, 2, \ldots \tag{2.6}$$

Several algorithms have been given to generate the Adomian polynomial rapidly in [26,33–36]. One of the convenient formula for Adomian polynomials is Rach's rule

$$A_n = \sum_{k=1}^{n} f^k(u_0) C(k, n), \quad n = 0, 1, 2, \ldots$$

where $C(k, n)$ denotes sums of all possible products of k components from $u_1, u_2, \ldots, u_{n-k+1}$, whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [35]. The list of first few Adomian polynomials for nonlinear function $f(u)$ is given below

$$\left.\begin{aligned} A_0 &= f(u_0), \\ A_1 &= u_1 f'(u_0), \\ A_2 &= u_2 f'(u_0) + \tfrac{1}{2} u_1^2 f''(u_0), \\ A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \tfrac{1}{6} u_1^3 f^{(3)}(u_0). \end{aligned}\right\}$$

where A_j depending on the solution components on $u_0, u_1, \ldots, u_j$.

By substituting the series (2.5) into (2.4), we get

$$\sum_{j=0}^{\infty} u_j(x) = \alpha_1 + \alpha_2 x + c_1 x^2 + c_2 x^3 + \mathcal{L}^{-1}[g(x)] + \mathcal{L}^{-1}\left[\sum_{j=0}^{\infty} A_j\right]. \tag{2.7}$$

Comparing both sides of (2.7), the ADM admits the following scheme:

$$\left.\begin{aligned} u_0(x) &= \alpha_1 + \alpha_2 x + c_1 x^2 + c_2 x^3 + \mathcal{L}^{-1}[g(x)], \\ u_j(x) &= \mathcal{L}^{-1}[A_{j-1}], \quad j \geq 1. \end{aligned}\right\} \tag{2.8}$$

Wazwaz [37] suggested a modified ADM (MADM) which is given by the scheme:

$$\left.\begin{aligned} u_0(x) &= \alpha_1, \\ u_1(x) &= \alpha_2 x + c_1 x^2 + c_2 x^3 + \mathcal{L}^{-1}[g(x)] + \mathcal{L}^{-1}[A_0], \\ u_j(x) &= \mathcal{L}^{-1}[A_{j-1}], \quad j \geq 2. \end{aligned}\right\} \tag{2.9}$$

The schemes (2.8) and (2.9) provide the complete determination of the components $u_n(x)$ of the solution $u(x)$. Hence, the n-term approximate series solution can be obtained by

$$\phi_n(x, c_1, c_2) = \sum_{j=0}^{n} u_j(x, c_1, c_2), \tag{2.10}$$

Note that the n-term truncated series solution $\phi_n(x, c_1, c_2)$ depends on the unknown constants c_1 and c_2. These unknown constants will be determined approximately by imposing the boundary condition at $x = b$ on $\phi_n(x, c_1, c_2)$, which leads a sequence of nonlinear system of equations as

$$\left.\begin{aligned} \phi_n(b, c_1, c_2) &= \alpha_3, \\ \phi_n'(b, c_1, c_2) &= \alpha_4, \quad n \geq 1. \end{aligned}\right\} \tag{2.11}$$

Solving the nonlinear system (2.11) for unknown constants c_1 and c_2, a huge amount of computational work is needed which combines some root-finding techniques such as Newton's methods. However, in some cases, these unknown constants c_1 and c_2 may not be determined uniquely. This is the main drawback of ADM or MADM for solving fourth-order BVPs. In order to avoid solving the nonlinear system (2.11) for unknowns constants, we propose an efficient algorithm based on ADM and the Green's function technique.

3 Decomposition method with Green's function

In this section, we propose an efficient recursive scheme for solving two-point fourth-order BVPs of the form (1.1), (1.2). To this end, we first consider the following homogeneous fourth-order BVPs as

$$\begin{cases} P^{(iv)}(x) = 0, \quad x \in J, \\ P(0) = \alpha_1, \;\; P'(0) = \alpha_2, \;\; P(b) = \alpha_3, \;\; P'(b) = \alpha_4. \end{cases} \tag{3.1}$$

The unique solution of BVP (3.1) is given by

$$P(x) = \alpha_1 + \alpha_2 x - \frac{(3\alpha_1 + 2b\alpha_2 - 3\alpha_3 + b\alpha_4)}{b^2} x^2 - \frac{(-2\alpha_1 - b\alpha_2 + 2\alpha_3 - b\alpha_4)}{b^3} x^3. \tag{3.2}$$

To construct Green's function of BVPs (1.1), (1.2), we consider the fourth-order linear BVP with homogeneous boundary conditions as

$$\begin{cases} u^{(iv)}(x) = F(x), \quad x \in J, \\ u(0) = u'(0) = u(b) = u'(b) = 0. \end{cases} \tag{3.3}$$

The Green's function of (3.3) can easily be constructed and it is given by

$$G(x,\xi)=\begin{cases} x^3\left(-\frac{1}{6}+\frac{\xi^2}{2b^2}-\frac{\xi^3}{3b^3}\right)+x^2\left(\frac{\xi}{2}-\frac{\xi^2}{b}+\frac{\xi^3}{2b^2}\right), & 0\le x\le \xi,\\ \xi^3\left(-\frac{1}{6}+\frac{x^2}{2b^2}-\frac{x^3}{3b^3}\right)+\xi^2\left(\frac{x}{2}-\frac{x^2}{b}+\frac{x^3}{2b^2}\right), & \xi\le x\le b. \end{cases} \tag{3.4}$$

It is easy to check that the function $G(x,\xi)$ satisfies all the properties of Green's function.

Using (3.2) and (3.4), we transform the original nonlinear fourth-order BVPs (1.1), (1.2) into the Fredholm integral equation as

$$u(x)=P(x)+\int_0^b G(x,\xi)\left[g(\xi)+f(\xi,u(\xi),u'(\xi),u''(\xi),u'''(\xi))\right]d\xi. \tag{3.5}$$

Note that the Eq. (3.5) does not involve any unknown coefficients to be determined.

We next decompose the solution $u(x)$ and the nonlinear function $f(x,u,u',u'',u''')$ by a series as

$$u(x)=\sum_{j=0}^{\infty}u_j(x) \quad\text{and}\quad f(x,u,u',u'',u''')=\sum_{j=0}^{\infty}A_j, \tag{3.6}$$

where A_j are Adomian's polynomials. In 2010, Duan [33,34] suggested several new efficient algorithms for rapid computer-generation of the Adomian's polynomials. Recently, Kalla [38] reported another more efficient programmable formula for Adomian's polynomials as

$$A_n=f(x,\psi_n,\psi_n',\psi_n'',\psi_n''')-\sum_{j=0}^{n-1}A_j \quad\text{or}\quad f(x,\psi_n,\psi_n',\psi_n'',\psi_n''')=\sum_{j=0}^{n}A_j \tag{3.7}$$

where $\psi_n=\sum_{j=0}^{n}u_j$ is the partial sum of the series solution $u=\sum_{j=0}^{\infty}u_j$.

Substituting the series (3.6) into (3.5), we obtain

$$\sum_{j=0}^{\infty}u_j(x)=P(x)+\int_0^b G(x,\xi)\left[g(\xi)+\sum_{j=0}^{\infty}A_j\right]d\xi. \tag{3.8}$$

Comparing both sides of (3.8), we have the following recursive scheme

$$\left.\begin{aligned} u_0(x)&=P(x)+\int_0^b G(x,\xi)g(\xi)d\xi,\\ u_j(x)&=\int_0^b G(x,\xi)A_{j-1}d\xi,\quad j\ge 1. \end{aligned}\right\} \tag{3.9}$$

We further modify the above recursive scheme (3.9) to get a more efficient and economic algorithm. To do this, the zeroth component $u_0(x)$ is divided into the sum of two parts, namely $g_0(x) + g_1(x)$, where $g_0(x) = \alpha_1$ and $g_1(x) = \alpha_2 x - \frac{(3\alpha_1+2b\alpha_2-3\alpha_3+b\alpha_4)}{b^2}x^2 - \frac{(-2\alpha_1-b\alpha_2+2\alpha_3-b\alpha_4)}{b^3}x^3 + \int_0^b G(x,\xi)g(\xi)d\xi$. The first part, $g_0(x)$, is kept in $u_0(x)$ and the rest part, $g_1(x)$, is added to $u_1(x)$. Consequently, we have a new modified recursive scheme as

$$\left.\begin{aligned} u_0(x) &= \alpha_1, \\ u_1(x) &= g_1(x) + \int_0^b G(x,\xi)[g(\xi) + A_0]d\xi, \\ u_j(x) &= \int_0^b G(x,\xi)A_{j-1}d\xi, \quad j \geq 2. \end{aligned}\right\} \tag{3.10}$$

The recursive schemes (3.9) and (3.10) provide the complete determination of solution components $u_j(x)$ of the solution $u(x)$. Hence, the n-term truncated series solution can be obtained as

$$\psi_n(x) = \sum_{j=0}^{n} u_j(x). \tag{3.11}$$

Note 3.1 *Unlike existing methods such as ADM or MADM, the proposed recursive schemes* (3.9) *and* (3.10) *do not involve any undetermined coefficients to be determined. In other words, it avoids solving a sequence of nonlinear algebraic or transcendental equations for the undetermined coefficients. It can be noted that the proposed recursive scheme* (3.9) *gives good approximate solution when the problem is linear or nonlinear of the form* $u^n, uu', u'^n \ldots$ *while the proposed modified scheme* (3.10) *is useful when the nonlinear function is of the form* e^u, $\ln u$, $\sin u$, $\cosh u \ldots$ *etc.*

4 Convergence analysis

In this section, we shall discuss the convergence analysis and the error estimation of the proposed recursive scheme (3.9) for nonlinear fourth-order BVPs (1.1), (1.2). Remark that many authors [39–41] have already discussed the convergence of the ADM for differential and integral equations. Let $\mathbb{X} = C^3[0,b]$ be the Banach space with the norm

$$\|u\| = \sum_{i=0}^{3} \max_{x \in J} \left|u^{(i)}(x)\right|, \quad u \in \mathbb{X}. \tag{4.1}$$

Now, we rewrite the integral Eq. (3.5) in the operator equation form as

$$u = \mathcal{N}(u), \tag{4.2}$$

where $\mathcal{N} : \mathbb{X} \rightarrow \mathbb{X}$ is a nonlinear operator given by

$$\mathcal{N}(u) = P(x) + \int_0^b G(x,\xi)[g(\xi) + f(\xi, u(\xi), u'(\xi), u''(\xi), u'''(\xi))]d\xi. \quad (4.3)$$

We next discuss the existence of the unique solution of the Eq. (4.2).

Theorem 4.1 (Banach Contraction Principle:) *Let $\mathbb{X}$ be the Banach space with the norm given by (4.1). Assuming that the f is uniformly Lipschitz continuous such that*

$$|f(x,u,u',u'',u''') - f(x,v,v',v'',v''')| \leq \sum_{i=0}^{3} K_i |u^{(i)} - v^{(i)}|,$$

where $K_i, i = 0,1,2,3$ are Lipschitz constants. Further, let δ be a constant defined as

$$\delta := 4MK,$$

where $M = \max_{i\in\{0,1,2,3\}} \left\{ \max_{x\in J} \int_0^b \left| \frac{\partial^i}{\partial x^i}(G(x,\xi)) \right| d\xi \right\}$ *and* $K = \max\{K_0, K_1, K_2, K_3\}$. *If $\delta < 1$, then the Eq. (4.2) has a unique solution u in $\mathbb{X}$.*

Proof For $i = 0,1,2,3$ and for any $u, v \in \mathbb{X}$, and using the Lipschitz continuity of f, we obtain

$$\begin{aligned} |(\mathcal{N}u - \mathcal{N}v)^{(i)}(x)| &\leq \int_0^b \left| \frac{\partial^i}{\partial x^i}(G(x,\xi)) \right| \big| f(\xi, u(\xi), u'(\xi), u''(\xi), u'''(\xi)) \\ &\quad - f(\xi, v(\xi), v'(\xi), v''(\xi), v'''(\xi)) \big| d\xi, \\ &\leq \max_{\xi\in J} \sum_{i=0}^{3} K_i |u^i(\xi) - v^i(\xi)| \times \left(\max_{x\in J} \int_0^b \left| \frac{\partial^i}{\partial x^i}(G(x,\xi)) \right| d\xi \right), \\ &\leq MK\|u - v\|, \end{aligned}$$

where $M = \max_{i\in\{0,1,2,3\}} \left\{ \max_{x\in J} \int_0^b \left| \frac{\partial^i}{\partial x^i}(G(x,\xi)) \right| d\xi \right\}$ and $K = \max\{K_0, K_1, K_2, K_3\}$.

Thus we have

$$|(\mathcal{N}u - \mathcal{N}v)^{(i)}(x)| \leq MK\|u - v\|, \quad \text{for } i = 0,1,2,3. \quad (4.4)$$

Using the estimates (4.4) for $i = 0, 1, 2, 3$, we obtain

$$\begin{aligned}\|\mathcal{N}u - \mathcal{N}v\| &= \sum_{i=0}^{3} \max_{x\in J} |(\mathcal{N}u - \mathcal{N}v)^{(i)}(x)| \le 4MK\|u - v\|,\\ &\le \delta\|u - v\|,\end{aligned} \tag{4.5}$$

where $\delta = 4MK$. If $\delta < 1$, then the nonlinear $\mathcal{N} : \mathbb{X} \to \mathbb{X}$ is contraction mapping and hence by the Banach contraction mapping theorem, the Eq. (4.2) has a unique solution in $\mathbb{X}$. □

We now rewrite the proposed method (3.9) in operator form as follows. Let $\{\psi_n = \sum_{j=0}^{n} u_j\}$ be a sequence of partial sums of the series solution $u = \sum_{j=0}^{\infty} u_j$. Using the recursive scheme (3.9) and the n-term series solution (3.11), we have

$$\begin{aligned}\psi_n = u_0 + \sum_{j=1}^{n} u_j &= P(x) + \sum_{j=1}^{n} \left[\int_0^b G(x,\xi)\left[g(\xi) + A_{j-1}\right] d\xi \right],\\ &= P(x) + \int_0^b G(x,\xi)\left[g(\xi) + \sum_{j=0}^{n-1} A_j \right] d\xi.\end{aligned} \tag{4.6}$$

Using (3.7) in (4.6), it follows that

$$\psi_n = P(x) + \int_0^b G(x,\xi)[g(\xi) + f(\xi, \psi_{n-1}, \psi'_{n-1}, \psi''_{n-1}, \psi'''_{n-1})]d\xi. \tag{4.7}$$

which is equivalent to the following operator equation

$$\psi_n = \mathcal{N}(\psi_{n-1}), \quad n = 1, 2, \ldots. \tag{4.8}$$

In the following theorem, we shall give the convergence of the sequence $\{\psi_n\}$ defined by (4.8) to the exact solution u of the Eq. (4.2).

Theorem 4.2 *Let $\mathcal{N}$ be the nonlinear operator defined by* (4.3) *is contractive, that is, $\|\mathcal{N}(u) - \mathcal{N}(u^*)\| \le \delta\|u - u^*\|,\ \forall\, u, u^* \in \mathbb{X}$ with $\delta < 1$ and $\|u_1\| < \infty$. Then the sequence $\{\psi_n\}$ of the partial sums given by* (4.8) *converges to the exact solution u of* (4.2).

Proof Using the relation (4.8) and the fact that nonlinear operator $\mathcal{N}$ is contractive, we have

$$\|\psi_{m+1} - \psi_m\| = \|\mathcal{N}(\psi_m) - \mathcal{N}(\psi_{m-1})\| \le \delta\|\psi_m - \psi_{m-1}\|.$$

Thus, we have

$$\|\psi_{m+1}-\psi_m\| \le \delta\|\psi_m-\psi_{m-1}\| \le \delta^2\|\psi_{m-1}-\psi_{m-2}\| \le \ldots \le \delta^m\|\psi_1-\psi_0\|.$$

For all $n, m \in \mathbb{N}$, with $n > m$, consider

$$\begin{aligned}\|\psi_n-\psi_m\| &= \|(\psi_n-\psi_{n-1})+(\psi_{n-1}-\psi_{n-2})+\cdots+(\psi_{m+1}-\psi_m)\|,\\ &\le \|\psi_n-\psi_{n-1}\|+\|\psi_{n-1}-\psi_{n-2}\|+\cdots+\|\psi_{m+1}-\psi_m\|,\\ &\le [\delta^{n-1}+\delta^{n-2}+\cdots+\delta^m]\|\psi_1-\psi_0\|,\\ &= \delta^m[1+\delta+\delta^2+\cdots+\delta^{n-m-1}]\|\psi_1-\psi_0\|,\\ &= \delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\|u_1\|.\end{aligned}$$

Since $0 < \delta < 1$, we have $(1-\delta^{n-m}) < 1$. It readily follows that

$$\|\psi_n-\psi_m\| \le \frac{\delta^m}{1-\delta}\|u_1\|. \tag{4.9}$$

Letting $m \to \infty$, we obtain

$$\|\psi_n-\psi_m\| \to 0.$$

Hence $\{\psi_n\}$ is cauchy sequence in $\mathbb{X}$. Hence there exits ψ in $\mathbb{X}$ such that $\lim_{n\to\infty}\psi_n = \psi$. Note that ψ is the exact solution of the Eq. (4.2) as

$$u = \sum_{n=0}^{\infty} u_n = \lim_{n\to\infty}\psi_n = \psi.$$

This completes the proof. □

Finally, we provide an error estimates for the truncated series solution in the following theorem.

Theorem 4.3 *Let u be the exact solution of the operator Eq.* (4.2). *Let ψ_m be the sequence of approximate series solutions defined by* (3.11). *Then there holds*

$$\|u-\psi_m\| \le \frac{4M\delta^m}{1-\delta}\max_{\xi\in J}|f(\xi, u_0, u_0', u_0'', u_0''')|.$$

Proof For any $n \ge m$, and using the estimate (4.9), we have

$$\|\psi_n-\psi_m\| \le \frac{\delta^m}{1-\delta}\|u_1\|. \tag{4.10}$$

Since $\lim_{n\to\infty}\psi_n = u$, fixing m and letting $n \to \infty$ in the estimate (4.10), we obtain

$$\|u-\psi_m\| \le \frac{\delta^m}{1-\delta}\sum_{i=0}^{3}\max_{x\in J}\left|u_1^{(i)}(x)\right|, \tag{4.11}$$

Since $u_1(x) = \int_0^b G(x,\xi)A_0 d\xi$ and $A_0 = f(\xi, u_0, u_0', u_0'', u_0''')$, we have

$$\sum_{i=0}^{3}\max_{x\in J}\left|u_1^{(i)}(x)\right| \le \sum_{i=0}^{3}\left(\max_{x\in J}\int_0^b\left|\frac{\partial^i}{\partial x^i}(G(x,\xi))\right|d\xi\right)\times\max_{\xi\in J}|f(\xi,u_0,u_0',u_0'',u_0''')|,$$
$$\le 4M\max_{\xi\in J}|f(\xi,u_0,u_0',u_0'',u_0''')|, \tag{4.12}$$

where $M = \max_{i\in\{0,1,2,3\}}\left\{\max_{x\in J}\int_0^b\left|\frac{\partial^i}{\partial x^i}(G(x,\xi))\right|d\xi\right\}$.

Combining the estimates (4.11) and (4.12), we obtain

$$\|u-\psi_m\| \le \frac{4M\delta^m}{(1-\delta)}\max_{\xi\in J}|f(\xi,u_0,u_0',u_0'',u_0''')|. \tag{4.13}$$

which completes the proof. □

5 Numerical results

To demonstrate the accuracy and applicability of the proposed method (3.9) and (3.10), we have solved several linear as well as nonlinear fourth-order BVPs. All numerical results obtained by the proposed method are compared with the known results.

Example 5.1 We first consider the following nonlinear fourth-order BVP

$$\left.\begin{array}{l} u^{(iv)}(x) = e^{-x}u^2(x),\ 0 < x < 1, \\ u(0) = u'(0) = 1,\ \ u(1) = u'(1) = e. \end{array}\right\} \tag{5.1}$$

The analytical solution is $u(x) = e^x$.

According to the proposed method (3.9), with $\alpha_1 = \alpha_2 = 1, \alpha_3 = \alpha_4 = e$ and $b = 1$. Consequently, we have the following recursive scheme as

$$\left.\begin{array}{l} u_0(x) = 1 + x + (-5+2e)x^2 + (3-e)x^3, \\ u_j(x) = \int_0^1 G(x,\xi)e^{-\xi}A_{j-1}d\xi,\quad j = 1, 2, \ldots, \end{array}\right\} \tag{5.2}$$

where the Green's function $G(x,\xi)$ is given by (3.4). Using the formula (2.6), we obtain the Adomian's polynomials for the nonlinear function $f(u(x)) = u^2(x)$ as follows

$$\left.\begin{aligned} A_0 &= u_0^2(x), \\ A_1 &= 2u_0(x)u_1(x), \\ A_2 &= 2u_0(x)u_2(x) + u_1^2(x), \\ &\vdots \\ A_n &= \textstyle\sum_{j=0}^{n} u_j(x)u_{n-j}(x). \end{aligned}\right\} \tag{5.3}$$

In view of (5.2) and (5.3), we obtain

$$\begin{aligned} u_0(x) &= 1 + x + 0.436564x^2 + 0.281718x^3 \\ u_1(x) &= 0.063189x^2 - 0.114671x^3 + 0.041666x^4 + 0.008333x^5 + 0.001036x^6 \\ &\quad + 0.000472x^7 - 0.000022x^8 + \cdots \\ u_2(x) &= -5.8207 \times 10^{-11}x + 0.000245x^2 - 0.000378x^3 - 2.008164 \times 10^{-9}x^4 \\ &\quad + 2.015440 \times 10^{-9}x^5 + 0.000351x^6 - 0.0002730x^7 + 0.000044x^8 + \cdots \\ &\vdots \end{aligned}$$

Hence, the truncated series solution is obtained as

$$\begin{aligned} \psi_2(x) &= 1 + x + 0.499999x^2 + 0.166669x^3 + 0.041666x^4 + 0.008333x^5 \\ &\quad + 0.001387x^6 + 0.0001993x^7 + 0.0000224x^8 + \cdots \end{aligned}$$

Note that all above components are computed by computer algebra system, such as 'MATHEMATICA'. For quantitative comparison, we define the absolute error functions as

$$E_n(x) = |\psi_n(x) - u(x)| \text{ and } e_n(x) = |\phi_n(x) - u(x)|,$$

where $u(x)$ is analytical solution, and $\psi_n(x)$ and $\phi_n(x)$ are n-term truncated series solutions obtained by the proposed method (3.9) and MADM (2.9), respectively. Table 1 shows the comparison between maximum absolute errors obtained by the proposed

Table 1 Comparison of the numerical results for Example 5.1

x	Proposed scheme		MADM	
	$\|\psi_1 - u\|$	$\|\psi_2 - u\|$	$\|\phi_1 - u\|$	$\|\phi_2 - u\|$
0.0	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00
0.2	6.8394E−06	3.5093E−08	9.8731E−04	2.5393E−04
0.4	1.6126E−05	8.2634E−08	2.4344E−03	6.5884E−04
0.6	1.6216E−05	8.2376E−08	2.6536E−03	7.6074E−04
0.8	6.9463E−06	3.4819E−08	1.2801E−03	3.9010E−04
1.0	0.0000E−00	0.0000E−00	0.0000E−00	1.5543E−14

method (3.9) and MADM (2.9). It is readily observed that the proposed method (3.9) provides not only better numerical results but also avoids solving difficult system of equations for unknown coefficients.

Example 5.2 Consider the following nonlinear fourth-order BVP [42]

$$\left.\begin{aligned} &u^{(iv)}(x) = g(x) + u^2(x), \quad 0 < x < 1, \\ &u(0) = u'(0) = 0, \quad u(1) = u'(1) = 1 \end{aligned}\right\} \tag{5.4}$$

where $g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48$. The analytical solution is $u(x) = x^5 - 2x^4 + 2x^2$.

According to the proposed method (3.9) with $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \alpha_4 = 1$ and $b = 1$. Consequently,

$$\left.\begin{aligned} &u_0(x) = 2x^2 - x^3 + \int_0^1 g(\xi)G(x,\xi)d\xi, \\ &u_j(x) = \int_0^1 G(x,\xi)A_{j-1}d\xi, \quad j \geq 1. \end{aligned}\right\} \tag{5.5}$$

Using the scheme (5.5) and the Adomian's polynomials (5.3), we calculate the components as follows

$$\begin{aligned} u_0(x) &= 1.99401x^2 + 0.007435x^3 - 2x^4 + x^5 - 0.002380x^8 + 0.001587x^{10} \\ &\quad - 0.000505x^{11} - 0.000336x^{12} + 0.000233x^{13} - 0.000041x^{14}, \\ u_1(x) &= 0.005979x^2 - 0.007419x^3 + 0.002366x^8 + 9.805402 \times 10^{-6}x^9 \\ &\quad - 0.001582x^{10} + 0.000499x^{11} + 0.000337x^{12} - 0.000233x^{13} \\ &\quad + 0.000041x^{14} - 1.080754 \times 10^{-9}x^{15} + \cdots \\ &\vdots \end{aligned}$$

Hence, the truncated series solution is obtained as

$$\begin{aligned} \psi_2(x) &= 2x^2 + 3.778011 \times 10^{-8}x^3 - 2x^4 + x^5 - 4.945957 \times 10^{-8}x^8 \\ &\quad + 5.013510 \times 10^{-8}x^9 - 1.529513 \times 10^{-9}x^{10} + \cdots \end{aligned}$$

Similarly, Table 2 shows the comparison of maximum absolute error obtained by the proposed method (3.9) and MADM (2.9). Once again, it is shown that the proposed method (3.9) gives better numerical results compared to MADM (2.9). Also note that the proposed approach avoids extra calculations for unknown constants.

Example 5.3 Consider the nonlinear fourth-order two-point BVPs [43]

$$\left.\begin{aligned} &u^{(iv)}(x) = u^2(x) + 1, \quad 0 < x < 2, \\ &u(0) = u'(0) = u(2) = u'(2) = 0. \end{aligned}\right\} \tag{5.6}$$

Table 2 Comparison of absolute error of Example 5.2

x	Proposed scheme			MADM		
	$\lvert\psi_1 - u\rvert$	$\lvert\psi_2 - u\rvert$	$\lvert\psi_3 - u\rvert$	$\lvert\phi_1 - u\rvert$	$\lvert\phi_2 - u\rvert$	$\lvert\phi_3 - u\rvert$
0.0	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00
0.2	3.4765E−07	8.1093E−10	2.0993E−12	1.8221E−04	3.5906E−05	5.4492E−09
0.4	8.9392E−07	2.0542E−09	5.2838E−12	5.0909E−04	1.0188E−04	4.3593E−08
0.6	9.9314E−07	2.2272E−09	5.6716E−12	6.5782E−04	1.3579E−04	1.4713E−07
0.8	4.6466E−07	1.0115E−09	2.5474E−12	3.9270E−04	8.5908E−05	3.4784E−07
1.0	0.0000E−00	0.0000E−00	0.0000E−00	2.6689E−13	5.5799E−13	6.3640E−07

We apply the proposed scheme (3.9) to (5.6), with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and $b = 2$, we have

$$\left.\begin{aligned} u_0(x) &= \int_0^2 G(x,\xi)d\xi, \\ u_j(x) &= \int_0^2 G(x,\xi)A_{j-1}d\xi, \quad j \geq 1. \end{aligned}\right\} \tag{5.7}$$

Using the recursive scheme (5.7) and the Adomian's polynomials (5.3), the components are obtained as

$$\begin{aligned} u_0(x) &= 0.166667x^2 - 0.166667x^3 + 0.0416667x^4 \\ u_1(x) &= 0.000160x^2 - 0.000117x^3 + 0.000016x^8 - 0.000018x^9 \\ &\quad + 8.267195 \times 10^{-6}x^{10} - \cdots \\ u_2(x) &= 3.604059 \times 10^{-7}x^2 - 2.635822 \times 10^{-7}x^3 + 3.181220 \times 10^{-8}x^8 \\ &\quad - 3.063397 \times 10^{-8}x^9 + 1.042733 \times 10^{-8}x^{10} - 1.237141 \times 10^{-9}x^{11} \\ &\quad + 2.294149 \times 10^{-10}x^{14} - \cdots \\ &\vdots \end{aligned}$$

Hence, the truncated series solution is obtained as

$$\begin{aligned} \psi_2(x) &= 0.166827x^2 - 0.166785x^3 + 0.0416667x^4 + 0.000016x^8 - 0.000018x^9 \\ &\quad + 8.277623 \times 10^{-6}x^{10} - 1.754884 \times 10^{-6}x^{11} + 1.461372 \times 10^{-7}x^{12} + \cdots \end{aligned}$$

Since the exact solution of this problem is not known. We check the accuracy of the proposed method (3.9) by means of the absolute residual error function

$$R_n(x) = |\psi_n^{(iv)}(x) - \psi_n^2(x) - 1|, \quad 0 < x < 2. \tag{5.8}$$

Table 3 Absolute residual error of Example 5.3

x	R_1	R_2	R_3	R_4	R_5
0.0	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00
0.2	5.9135E−08	1.6300E−10	5.0947E−13	1.7131E−15	6.0478E−18
0.4	6.1934E−07	1.7233E−09	5.4082E−12	1.8227E−14	6.4618E−17
0.6	1.9096E−06	5.3564E−09	1.6868E−11	5.6960E−14	2.0155E−16
0.8	3.3403E−06	9.4204E−09	2.9735E−11	1.0054E−13	3.5518E−16
1.0	3.9671E−06	1.1209E−08	3.5413E−11	1.1979E−13	4.2392E−16
1.2	3.3403E−06	9.4204E−09	2.9735E−11	1.0054E−13	3.5605E−16
1.4	1.9096E−06	5.3564E−09	1.6868E−11	5.6967E−14	2.0903E−16
1.6	6.1934E−07	1.7233E−09	5.4082E−12	1.8176E−14	1.3986E−17
1.8	5.9135E−08	1.6300E−10	5.0954E−13	1.7837E−15	7.6653E−17
2.0	0.0000E−00	1.6517E−20	0.0000E−00	1.6611E−20	1.6609E−20

Here the n-term truncated series solution $\psi_n(x)$ is used in place of $u(x)$ in order to check the convergence of ψ_n to $u(x)$, since $\psi_n \to u$ as $n \to \infty$. Table 3 shows the numerical results of absolute residual error R_n, for $n = 1, 2, 3, 4, 5$. It can be seen from the numerical results in Table 3 that the proposed method (3.9) gives the approximate series solution which rapidly converges to the exact solution as n becomes very large.

Example 5.4 Consider the following fourth-order BVP [7]

$$\left.\begin{aligned} &u^{(iv)}(x) = -6e^{-4u(x)}, \quad 0 < x < 4-e, \\ &u(0) = 1, \; u'(0) = \frac{1}{e}, \quad u(4-e) = \ln 4, \; u'(4-e) = \frac{1}{4}. \end{aligned}\right\} \tag{5.9}$$

The exact solution is $u(x) = \ln(e + x)$.

According to the proposed method (3.10) with $\alpha_1 = 1$, $\alpha_2 = \frac{1}{e}$, $\alpha_3 = \ln 4$, $\alpha_4 = \frac{1}{4}$, and $b = 4 - e$. Consequently, we have the following scheme as

$$\left.\begin{aligned} &u_0(x) = 1, \\ &u_1(x) = 0.3678794x - 0.0636607x^2 + 0.0091938x^3 + \int_0^{4-e} G(x,\xi)A_0 d\xi, \\ &u_j(x) = \int_0^{4-e} G(x,\xi)A_{j-1} d\xi, \quad j \geq 1. \end{aligned}\right\} \tag{5.10}$$

The Adomian's polynomials for $f(u(x)) = -6e^{-4u(x)}$ are calculated as:

$$\left.\begin{aligned} &A_0 = -6e^{-4u_0(x)} \\ &A_1 = 24u_1(x)e^{-4u_0(x)} \\ &A_2 = \left(-48u_1^2(x) + 24u_2(x)\right) e^{-4u_0(x)} \\ &A_3 = \left(64u_1^3(x) - 96u_1(x)u_2(x) + 24u_3(x)\right) e^{-4u_0(x)} \\ &\vdots \end{aligned}\right\} \tag{5.11}$$

Table 4 Numerical results for Example 5.4

x	Proposed method				
	$\|\psi_1 - u\|$	$\|\psi_2 - u\|$	$\|\psi_3 - u\|$	$\|\psi_4 - u\|$	$\|\psi_5 - u\|$
0.0	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00
0.2	1.0633E−04	4.9684E−05	1.6216E−05	4.1012E−06	8.2922E−07
0.4	2.9725E−04	1.4553E−04	4.8810E−05	1.2574E−05	2.5815E−06
0.6	4.1755E−04	21.513E−04	7.4716E−05	1.9762E−05	4.1536E−06
0.8	3.8488E−04	2.0906E−04	7.5644E−05	2.0714E−05	4.5015E−06
1.0	2.1259E−04	1.2175E−04	4.6096E−05	1.3163E−05	2.9868E−06
1.2	2.6523E−05	1.6004E−05	6.3557E−06	1.9026E−06	4.5393E−07
4−e	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00

Making use of (5.10) and (5.11), it follows

$$\begin{aligned} u_0(x) &= 1 \\ u_1(x) &= 0.367879x - 0.071183x^2 + 0.020931x^3 - 0.004578x^4, \\ u_2(x) &= 0.005096x^2 - 0.006032x^3 + 0.001347x^5 - 0.000086x^6 + 0.000010x^7 \\ &\quad - 1.1980808 \times 10^{-6}x^8 + 5.421010 \times 10^{-20}x^{10}, \\ &\vdots \end{aligned}$$

Hence, the truncated series solution is obtained as

$$\begin{aligned} \psi_2(x) &= 1 + 0.367879x - 0.066086x^2 + 0.014899x^3 - 0.004578x^4 + 0.001347x^5 \\ &\quad - 0.000086x^6 + 0.000010x^7 - 1.198080 \times 10^{-6}x^8 + \cdots \end{aligned}$$

Tables 4 and 5 show the comparison of the numerical results obtained by the proposed method (3.10) and MADM (2.9). Once again, we have shown that the proposed method (3.10) gives much better numerical results compared to MADM (2.9). Also our scheme avoids solving difficult system of equations for unknown constants.

Example 5.5 Consider the following fourth-order BVP [10]

$$\left.\begin{array}{l} u^{(iv)}(x) = u(x) + u''(x) + e^x(x-3), \quad 0 < x < 1, \\ u(0) = 1, u'(0) = 0, \quad u(1) = 0, \quad u'(1) = -e. \end{array}\right\} \tag{5.12}$$

The exact solution is $u(x) = (1-x)e^x$.

Table 5 Numerical results for Example 5.4

x	MADM				
	$\|\phi_1 - u\|$	$\|\phi_2 - u\|$	$\|\phi_3 - u\|$	$\|\phi_4 - u\|$	$\|\phi_5 - u\|$
0.0	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00
0.2	1.0633E−02	4.9874E−03	1.6336E−03	4.1883E−04	8.7523E−05
0.4	2.9725E−02	1.4600E−02	4.9088E−03	1.2786E−03	2.6994E−04
0.6	4.1755E−02	2.1563E−02	7.4954E−03	1.9971E−03	4.2833E−04
0.8	3.8488E−02	2.0929E−02	7.5633E−03	2.0753E−03	4.5510E−04
1.0	2.1259E−02	1.2171E−02	4.5895E−03	1.3038E−03	2.9409E−04
1.2	2.6523E−03	1.5969E−03	6.2956E−04	1.8581E−04	4.3270E−05
4−e	0.0000E−00	2.2200E−16	6.8800E−15	2.4400E−15	2.2200E−16

According to the proposed method (3.9) with $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = -e$ and $b = 1$. Consequently, we have the following scheme as

$$\left.\begin{aligned} u_0(x) &= 1 + (-3+e)x^2 + (2-e)x^3 + \int_0^1 e^{\xi}(\xi - 3)G(x,\xi)d\xi, \\ u_j(x) &= \int_0^1 G(x,\xi)\left[u_{j-1}(\xi) + u''_{j-1}(\xi)\right]d\xi, \quad j \geq 1. \end{aligned}\right\} \tag{5.13}$$

Using the recursive scheme (5.13), we can obtain the components as

$$\begin{aligned} u_0(x) &= 8 - 7e^x + \left(6 + e^x\right)x + 2.05595x^2 + 0.253745x^3, \\ u_1(x) &= 20\left(1 - e^x\right) + (18 + 2e^x)x + 7.94309x^2 + 2.41262x^3 + 0.504662x^4 \\ &\quad + 0.0626873x^5 + 0.00571096x^6 + 0.000302078x^7, \\ &\ \vdots \end{aligned}$$

Hence, the truncated series solution is obtained as

$$\begin{aligned} \psi_2(x) &= 80 - 79e^x + 72x + 7e^x x - 804.296x^2 - 9.33334x^3 - 1.99992x^4 \\ &\quad - 0.333318x^5 - 0.0445972x^6 - 0.0046668x^7 - 0.000402376x^8 \\ &\quad - 0.000024x^9 - 1.13312 \times 10^{-6}x^{10} - 3.814113 \times 10^{-8}x^{11} + \cdots \end{aligned}$$

Table 6 shows the comparison of maximum absolute error obtained by the proposed method (3.9) and MADM (2.9). It can clearly be seen from these results that the proposed scheme (3.9) gives better numerical results compared to MADM (2.9).

Remark It can be seen from the numerical results of all five examples discussed in this section that only two terms are sufficient for obtaining good approximations to the exact solution.

Table 6 Maximum absolute error estimate of Example 5.5

x	Proposed scheme			MADM		
	$\|\psi_1 - u\|$	$\|\psi_2 - u\|$	$\|\psi_3 - u\|$	$\|\phi_1 - u\|$	$\|\phi_2 - u\|$	$\|\phi_3 - u\|$
0.0	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00	0.0000E−00
0.2	3.4867E−05	8.2085E−07	1.9470E−08	2.6734E−03	1.8398E−04	5.0234E−06
0.4	9.4007E−05	2.2181E−06	5.2263E−08	6.4531E−03	4.8317E−04	1.3828E−05
0.6	1.0015E−04	2.2939E−06	5.3189E−08	6.9415E−03	5.6954E−04	1.7483E−05
0.8	4.1852E−05	9.0199E−07	2.0439E−08	3.3275E−03	3.0057E−04	1.0131E−05
1.0	0.0000E−00	0.0000E−00	0.0000E−00	8.8876E−16	5.8845E−15	4.6145E−15

6 Conclusion

In this work, we have proposed a new efficient approach for obtaining approximate series solution of the fourth-order two-point BVPs. The simplicity, efficiency and reliability of the proposed recursive schemes (3.9) and (3.10) have been examined by solving five fourth-order BVPs. The accuracy of the numerical results indicates that the proposed method is well suited for the approximate solutions of such BVPs. It has also been shown that only two-terms are sufficient to obtain a comparable approximate solutions. Unlike the existing methods such as the ADM or MADM, the proposed schemes (3.9) and (3.10) do not require any computation of unknown constants, and provides much better numerical results. Furthermore, by comparing the proposed method with ADM or MADM, we have also shown that the proposed method gives not only better numerical results but also avoids solving a sequence of growingly higher order polynomial or difficult transcendental system of equations for obtaining unknown constants. Unlike finite different method, cubic spline method or any other discrete methods, the proposed method does not require any linearization, perturbation or discretization of variables. Finally, we have discussed the convergence and error analysis of the proposed method (3.9).

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